Advanced Quantum Field Theory – Syllabus AY 2018/19

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Oral examen: topics between square brackets are excluded from the examen. Only main results, without details of proofs, are required for subjects between round brackets. The proposed exercises, see separate file, are part of the examen and serve as personal check of your comprehension.

1) Perturbative and non-perturbative aspects of quantum field theories. Operatorial approach and functional integral approach. (Haag’s theorem and non-existence of the interaction picture [EF]. Dyson’s argument for QED [ZI: 15.5], non-Borel summability of the perturbative series, Lautrup’s renormalons [L1], asymptotic series. Triviality-problem of the $\lambda\varphi^4$-theory. Wightman’s axiomatic approach and the reconstruction theorem [SW], [S4]. From Schwinger’s $SU(2)$ model to ’t Hooft’s proof of the renormalizability of YM theories).

2) Symmetries and quantum-mechanically consistent interactions. The paradigm of canonical quantization. Unitary implementation of global symmetries in quantum theories. Invariances of the $S$-matrix. Poincaré group and internal symmetries. Coleman-Mandula theorem [S1: 2.5]. Representations of the Lorentz group and spin. [Duality between scalar fields and antisymmetric potentials]. General properties of theories with particles of spin $0 \leq s \leq 1$, $3/2 \leq s \leq 2$ and $2 < s$.


5) Functional integral technics. The functionals $Z$, $W$ and $\Gamma$. The Euclidean space as the appropriate framework. Conventions for correlation functions and Fourier transforms. Translation invariance. Background field method and classical limit $\Gamma \to I$. Linear classical symmetries and their quantum implementation: proof of Furry’s theorem, mass protection through chiral symmetry, flavor conservation. Ward identity in QED as consequence of local gauge invariance: form of divergent counterterms and renormalizability, transversality of the photon vacuum polarization $\Gamma^{\mu\nu}$ [and relation with current conservation; relation between $\Gamma^{\mu\nu}$ and the correlation function $\langle 0 | T j^{\mu} j^{\nu} | 0 \rangle$ [L2: 103, 104]]; proof of the identities $Z_{1\nu} = Z_{2\nu}$. (Functional determinants for real and complex, commuting and anticommuting, scalar fields, and for spinor fields. One-loop effective action as a determinant. Effective scalar potential. Coleman-Weinberg mechanism for $\lambda\varphi^4$-theory and radiative
spontaneous symmetry breaking; dimensional transmutation). [Remarks on the mechanism in scalar QED [WII: 16.2], [IZ: 9.2.2, 11.2.2]].

6) Perturbative methods and renormalizability. Derivation of the Feynman rules from the Lagrangian of a generic quantum field theory. The concept of strictly renormalizable theories. Application to scalar QED: ultraviolet divergences of the four-point scalar correlation function. (Proof of locality of the one-loop divergences in a generic local quantum field theory. Criterion for the determination of the superficially divergent correlation functions. Determination of all power-counting renormalizable couplings in a generic $D$-dimensional space-time). Non-renormalizable theories and super-renormalizable theories [IZ: 8.1.3]. The $\lambda \phi^3$-theory in $D = 6$ as a limit theory. [The Gross-Neveu model in $D = 3$ as a non-perturbatively consistent theory, although non-renormalizable by power counting].

7) $\lambda \phi^3$-model in $D = 6$ and higher-order renormalizability. Superficial divergences and sub-divergences. Explicit one-loop renormalization: computation of $Z$, $Z_1$, the $\beta$-function, and the anomalous dimension $\gamma$; running coupling constant and asymptotic freedom [S2: 14, 16]. (Explicit solution of the Callan-Symanzik equation and the phenomenon of dimensional transmutation). Determination of one-loop counterterms. (Two-loop diagrams for the propagator and the three-point function: local and non-local divergences, the problem of nested and overlapping divergences. Proof of the cancelation of all non-local divergences at two loops, and the role of the subtraction of one-loop divergences. Subtraction of remnant superficial local divergences at two loops [C: 5.2]). [$n$-point correlation functions with $n > 3$ and higher loops].

8) Quantization of YM theories. Negative norm states in YM theories and the problem of manifest covariance. The gauge-fixing problem in the functional integral approach. (Canonical quantization in the axial gauge, and the corresponding expression of gauge-invariant correlation functions in the functional integral approach from first principles [WI: 9.2], [WII: 15.4], [IZ: 12.2.1]). Gauge invariance of the functional measure over matter fields and YM potentials [WII: 15.4]. Faddeev-Popov quantization method: functional $\delta$-function identities and FP determinant [P: 3.1]. Independence of gauge-invariant correlation functions of the gauge-fixing. The Lautrup weighting functionals $H[b]$. The Feynman-Lorenz $\lambda$-gauge. (Weinberg’s theorem). Connection of the FP method with the axial gauge and first principles [WII: 15.5]. FP determinant and ghost fields, anticommutation statistics, ghost number conservation. The Nakanishi-Lautrup auxiliary fields $B^a$ [WII: 15.6].

9) BRST symmetry and physical states. Missing gauge invariance of the FP action. BRST symmetry as the key ingredient ensuring the strict renormalizability of a YM theory and the decoupling of non-physical states. BRST transformations as a global one-parameter anticommuting symmetry group. Nilpotency of the transformations and invariance of the action. The FP action as a trivial cocycle. On-shell nilpotency in the absence of auxiliary fields. The conserved BRST charge $Q$ in canonical quantization and its properties. Requirement of gauge-fixing independence of correlation functions of physical operators and the Kugo-Ojima condition for physical states. $Q$ commutes with the $S$-matrix. The physical positive-definite Hilbert space $\mathcal{H}$ as the cohomology of $Q$. (Canonical quantization of the asymptotic fields and their commutation relations with $Q$. Proof of the absence of ghosts and non-physical gauge bosons in $\mathcal{H}$ [WII: 15.7]. Conserved BRST current. Explicit expression of $Q$ and relation with the Gupta-Bleuler condition in QED).

10) Slavnov-Taylor identities. The problem of strict renormalizability of the FP-gauged-fixed YM action. Non-linearity of the BRST transformations, and the external currents $K$. ST identity for the functionals $S$ and $\Gamma$. (Schwinger-Dyson equations [IZ: 10.1]). ST identities for the reduced functionals $\tilde{S}$ and $\tilde{\Gamma}$. Non-propagation of the gauge-fixing terms. Dependence of $\tilde{\Gamma}$ on $K_{\mu}^a$ and $\bar{C}^a$. Transversality
of the exact gluon vacuum polarization $\Gamma_{\mu\nu}^{ab}(p)$. [Relation between the transversality of $\Gamma_{\mu\nu}^{ab}(p)$ and the quantum conservation of the color currents $j^a\mu$.] Correlation functions as invariant tensor fields. Preservation of the ST identities under renormalization: (general solution of the equations $\tilde{S} \ast \tilde{S} = 0$ and $\tilde{S} \ast F + F \ast \tilde{S} = 0$), and the structure of the divergent counterterms. Strict renormalizability of non-abelian gauge theories at all orders in perturbation theory. Validity of the ST identities for the renormalized theory [WII: 16.4, 17.1, 17.2], [IZ: 12.4.3].

11) Perturbative one-loop analysis of non-abelian gauge theories. Derivation of the Feynman rules [IZ: 12.2.3]. ST identities for the ratios between renormalization constants [IZ: 12.3.4]. Renormalization and transversality of the gluon vacuum polarization: (explicit evaluation of the diagrams with gluon loops, fermion loops, and ghost loops). The role of the ghost fields w.r.t. preservation of unitarity. One-loop divergences of the fermion two-point function and of the fermion-fermion-gluon correlation function. (Computation of the renormalization constants $Z_3$, $Z_\psi$, $Z_{1\psi}$). Determination of the $\beta$-function. Running coupling constant, asymptotic freedom, and dimensional transmutation in QCD. The scale $\Lambda_{\text{QCD}}$. Comparison between the theories QED, QCD, $\lambda \varphi^4$ in $D = 4$, and $\lambda \varphi^3$ in $D = 6$ [R1: 9.8], [R2: 8.6, 8.8], [S2: 73], [PS: 16.5]. (Contribution to the $\beta$-function of scalar fields. $\beta$-functions and running of the coupling constants $\alpha_1$, $\alpha_2$ and $\alpha_3$ in the Standard Model of elementary particles. $SU(5)$-Grand Unification and merging of the coupling constants in the minimal supersymmetric Standard Model. $N = 4$ Super-YM theory as a perturbatively finite theory with all-order vanishing $\beta$-function).

12) BRST quantization of gauge theories with spontaneous symmetry breaking. [Goldstone theorem and Higgs mechanism revisited, massive gauge bosons. The FP method applied to the 't Hooft gauge: massive ghost fields. BRST symmetry and Kugo-Ojima condition on physical states. Decoupling of ghosts and goldstone bosons].


14) Instantons. Perturbative approach to the functional integral. Field configurations connected continuously to the absolute minimum of the Euclidean action. YM theories in Euclidean space-time. Pure-gauge YM connections. Relevant configurations and finiteness of the action. Classification of YM connections through the Pontryagin invariant $n$. The homotopy group $\pi_3(G)$ of a Lie group $G$. (The Maurer-Cartan invariant). Instantons as absolute minima of the action at fixed $n$. Instantons for $G = SU(2)$ relative to $n = 1$. [Moduli as collective coordinates. The case $G = SU(N)$]. Non-perturbative character of the weighting factor $e^{-8\pi^2n/g^2}$. $\vartheta$-vacua and the strong CP problem. [The axial $U(1)$ problem]. [W2: 23.4, 23.5 23.6], [VV].

Main textbooks:

References


[S4] F. Strocchi, *Selected topics on the general properties of Quantum Field Theory*.


From quantum field theory we know that field theories contain loop diagrams with ultra-violet ($k^2 \rightarrow \infty$) or the infrared ($k^2 \rightarrow 0$) divergences. One example of a UV-divergent integral is, \[ \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)((k - q)^2 - m^2)} \]

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**References**

- Peskin-Schroder, Chapter 19
- Yadurain: Theory of Quark and Gluon Interactions, Ch. 7.2.1 Chiral Symmetry of QCD.
- Srednicki, Quantum Field Theory, (Cambridge University Press, 2007) Another good general textbook, organised differently.
- Ohlsson T. Quantum physics and special relativity theory were two of the greatest breakthroughs in physics during the twentieth century and contributed to paradigm shifts in physics. This book combines these two discoveries to provide a complete description of the fundamentals of relativistic quantum physics, guiding the reader effortlessly from relativistic quantum mechanics to basic quantum field theory. The book gives a thorough and detailed treatment of the subject, beginning with the classification of particles, the Klein-Gordon equation, and the Dirac equation for fermions.